Journal of Engineering Science, Vol. 14, 71–99, 2018

Mathematical Implications of Hydrogeological Conditions: Modelling for Aquifer Parameters Determination

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Published online: 15 April 2018
To cite this article: Sheriffat Taiwo Okusaga, Mustapha Adewale Usman and Niyi Olaonipekun Adebisi. (2018). Mathematical implications of hydrogeological conditions: Modelling for aquifer parameters determination. *Journal of Engineering Science*, 14: 71–99, https://doi.org/10.21315/jes2018.14.6.
To link to this article: https://doi.org/10.21315/jes2018.14.6

Abstract: *Formulations of natural hydrogeological phenomena have been derived, from experimentation and observation of Darcy fundamental law. Mathematical methods are also been applied to expand on these formulations, but the management of complexities which relate to subsurface heterogeneity is yet to be developed into better models. This study employs a thorough approach to modelling flow in an aquifer (a porous medium) with phenomenological parameters such as Transmissibility and Storage Coefficient on the basis of mathematical arguments. An overview of the essential components of mathematical background and a basic working knowledge of groundwater flow is presented. Equations obtained through the modelling process were used to separate variables and solve hypothetical problems. Based on finite-difference conservative scheme the essential components of aquifer were determined by solving problem of groundwater flow in a porous medium.*

Keywords: Hydrogeological, mathematical, Darcy, aquifer, storage and transmissibility

1. INTRODUCTION

Groundwater is a component of the hydrologic cycle (Figure 1) which begins its journey as rainfall (precipitated water), percolates vertically downwards even to a greater depth through a geological formation, which could be soil and rock. Groundwater occurs in the cracks, joints and faults within a crystalline rock mass, and within the pore spaces of a sedimentary rock, such as sandstone. It is moves through and saturates a geologic formation (an aquifer/a porous medium), which could be soil or rock. Water attains a level in the ground which all available and interconnected spaces in the soil or rock are saturated. This is called water table. The water table is not flat but slope the hills down the valley sides towards the river.

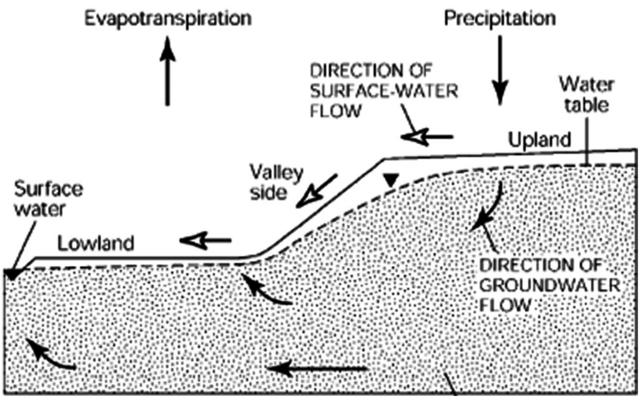

Figure 1: The hydrological cycle and groundwater flow (after).[1]

It is important to note that groundwater usually contains higher concentrations of natural dissolved materials than surface water. The materials dissolved in the water usually reflect the composition and solubility of the earth materials (soil or rock) that the groundwater is in contact with and time that it has been in the subsurface. Groundwater also plays a crucial role in sustaining rivers and streams, particularly during droughts when it becomes a valuable buffer. Many ecosystems including some of our most iconic depend on groundwater. Groundwater is a finite resource and aquifers can become depleted when extraction rates exceed replenishment or "recharge" rates, like surface water, groundwater can become polluted or contaminated.

The following are the importance of ground water:

1. It supplies drinking water, it helps grows our food (irrigation to grow crops).
2. It is an important component in many industrial processes.
3. It is a source of recharge for lakes, rivers and wetlands.

The pioneering solution of flow equations for aquifers was based on the analogy between flow of water in an aquifer and flow of heat in a thermal conductor.[2] The equation was adapted from heat transfer literature for two-dimensional radial flow to a point source in an infinite, homogeneous aquifer. Another extension to the traditional form of Darcy law is the account for transitional flow between boundaries[3] in comparing the solution with the Theis solutions[4]. However, the resulting answers are almost the same as these are graphical method of solutions and are in agreement with this work.

Theis solution was also applied to steady state radial flow which gives rise to a pumping well. It comes about from application of Darcy law to cylindrical shell control volume where the background he is the hydraulic head and $h–h_0$ is the drawdown at the radial distance from the pumping well. Jacob observed that after pumping well had been running for sometimes, higher values of the infinite series become very small and could be approximated.

In spite of the tremendous advancement in the application of modelling groundwater flow, little work has been carried out in the area of groundwater flow and storage. The solution of a "force convention" problem has been illustrated in one dimension.[5] The domain under study is an aquitard through which groundwater movement is vertical. This is because the fluids at depth are hotter than fluids near surface. The hotter fluids are less dense and lighter than the cooler fluids and tend to rise. A summary of these is among the several mechanisms that drive the flow of groundwater in sedimentary basins, topography ranks as one of the most important driving forces.[6]

The Bausinesq's approximation is considered, and treatment of the coupled equation of heat and fluid flow for the assumption of a constant fluid density everywhere expect for the buoyancy driving force.[7] The problem that arises in groundwater flow through a model in a fractured rock has also been a subject of research.[8] They attempt to fit a conventional radial flow model to observe the drawdown at early time underestimate and later time overestimate. Moreover, there are many fractured rocks where the flow of groundwater does not fit the application of Darcy law, the geometry of which differs completely from porous media.

Investigation suggested that the flow is influenced by the geometry of the bedding parallel fractures, a feature that the model cannot account for.[9,10] It is therefore possible that the equation may not be applicable to flow in porous media rather than a fractured aquifer. Rätz and It has been shown that the proposed Darcy's law relies on experimental results obtained from the flow of water through a one-dimensional sand column, the geometry of which differs completely from that of a fracture.[11,12] The derivation of a generalised groundwater flow equation is an indication that the contribution is to investigate the possibility of the development of a three-dimensional model for groundwater flow equation. Mathematical

Modelling is a tool that can be used for understanding of a groundwater system and its behaviour so as to predict its future response.

2. BACKGROUND OF STUDY AND PROBLEM FORMATION

The use of aquifers is increasing as both a source of water supply and a medium for storing various hazardous waters. As this usage expands, our knowledge of groundwater systems must also expand.

In general, specifying the flow domain is a major question in formulating the groundwater flow problem. The governing equation is the groundwater flow equation, expressed as:

$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{yy}\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_{zz}\frac{\partial h}{\partial z}\right)=S_s\frac{\partial h}{\partial t}-R \quad (1)$$

where k_{xx}, k_{yy}, k_{zz} are values of hydraulic conductivity along the x, y and z coordinates axes (L/T), h is the potentiometric head (L), R is a volumetric flux per unit volume representing sources or sinks of water (T^{-1}), Ss is the Specific Storage (L^{-1}) while T is the time (T).

The boundary conditions specify the flow equations at the boundary of the domain.[13] The two common boundary conditions are:

1. specified head and
2. specified flow.

The initial condition defines the spatial distribution of hydraulic head everywhere in the flow domain at the initial time.

3. MATHEMATICAL AND MODEL FORMULATIONS

3.1 Mass Balance Theory

A balanced mass must be performed along with Darcy law to arrive at the groundwater flow equation. This balance is analogous to the energy balance used in heat transfer to arrive at the heat equation. It is simply a statement accounting that for a given control volume, aside from sources or sinks mass can neither be created nor destroyed. It follows that for a given increment of time at the difference between the mass flowing across the boundaries and the sources within the volume is the change in storage.[14]

Therefore, the excess of in flow over outflow during a short time interval through the surface of the control volumes that are perpendicular to the *xy*- direction may be expressed as follows:

$$\frac{\Delta M_{stor}}{\Delta t} = \frac{M_{in}}{\Delta t} - \frac{M_{out}}{\Delta t} - \frac{M_{gen}}{\Delta t} \tag{2}$$

3.2 Fluid Mass Conservation

A fluid flow fluid through porous media can be described by differential equations.[15] Since the flow is a function of several variables, it can be appropriately described by partial differential equations in which the special coordinates *x*, *y* and *z* and time (*t*) are the independent variables. In depriving the equations, the law of mass conservation and energy are employed.

The law of fluid mass conservation also stated that in a flow system, fluid mass is neither created nor destroyed, which translates the above statement into a mathematical expression.[16,17] In this case, it is convenient to considered "control volume" within a flow region. The principle of conservation of fluid mass can now be stated as follows: The rate at which fluid mass enters the control volume minus the rate at which fluid mass leaves the control volume is equal to the rate at which fluid mass is accumulating in the elemental control volume.[18]

3.3 Control Volume Approach

Considering a very small part of an aquifer shown in Figure 2 as control volume, the general conservation statements, which incorporate boundary condition at the phreatic surfaces is still applicable.[19] The shape of the control volume is arbitrary with a definite volume, fixed in space and its boundaries are called control surfaces. The amount and identity of matter in the control volume may with time but the shape and position of this volume remain fixed.

The continuity equation or equation of mass conservation is obtained by considering a small control volume in Cartesian coordinates (*x*,*y*,*z*) of dimensions and , which are parallel to *x*, *y*, *z* coordinate around the point *P*(*x*, *y*, *z*) in a porous medium domain. The vector (*V*) with components V_x, V_y, V_z in the *x*, *y*, *z* directions denote the mass flux of a fluid of density (ρ). The area faces normal to the *x*-axis is *dy dz*, the area of the faces normal to the *y*-axis is *dx dz*, and the area of the faces normal to the *z*-axis is *dx dy*.

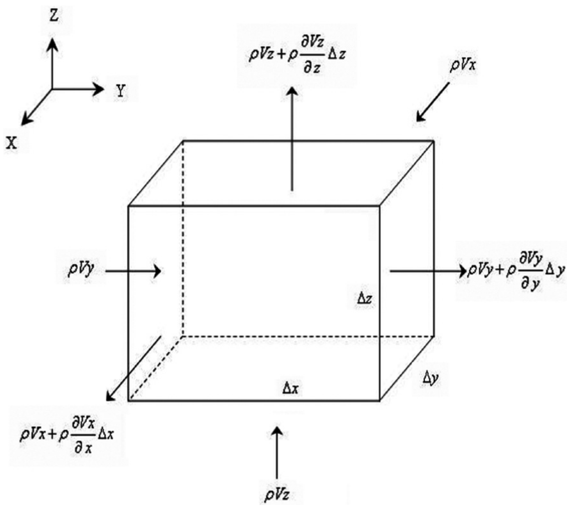

Figure 2: Control volume of dimensions, ∂x, ∂y, ∂z and centers $p(x, y, z)$.

Assuming the aquifer is homogeneous and isotropic, the fluid moves only in one direction through the control volume. The actual motion can be subdivided on the basis of the components flow parallel to the three principal axes. For this analysis, we assumed the main characteristic flow in an aquifer is that flow is essentially horizontal. This is true, in a confined horizontal homogeneous, isotropic aquifer with fully penetrating wells. This is in order when the thicknesses of the aquifer vary in such a way that is much smaller than the aquifer thickness itself. This assumption fails in regions where the flow has a vertical component. However, in view of the fact that the thickness of the aquifer is much less than the horizontal length involved, the assumption of essentially horizontal flow may be considered a good approximation.

Therefore, under certain conditions, instead of considering the flow in three dimensional with $h(x, y, z, t)$ we may treat the problem in terms of an average head, $h = h(x, y, t)$ if the flow is two dimensional in the horizontal xy- plane. All terms from control volume involving first and second derivatives with respect to z- axis vanish.

The general continuity equation is:

$$q = av \tag{3}$$

where q is the flow rate, volume/time (L^3T^{-1}), a is cross sectional area perpendicular to the flow, (L^2) while v is the flow velocity, length/time (LT^{-1}).

If Q is the total flow rate, then the mass flux $V_x = {}_w q_x$ is the portion parallel to the x- axis, where p_w is the water density and $V_y = {}_w q_y$ in the y- direction respectively.

4. DERIVATION OF GROUNDWATER FLOW EQUATION

The groundwater flow equation is a mathematical relationship, which describes the flow of water through a geological formation that can store and transmit water. The steady state flow of groundwater is described by a form of the Laplace-equation, which is a form of potential flow and has analogs in numerous fields. The groundwater flow equation is often derived for a small Representative Elemental Volume (REV) as illustrated in Figure 2 where the properties of the medium are assumed to be effectively constant. A mass balance is done on the water flowing in and out of this small volume, the flux terms in the relationship being expressed in terms of head by using the constitutive equation (Darcy law), which requires that the flow is slow.

4.1 Flow in x- Direction

Volume into face $x = q_x A_x$
where $A_x = dydz$
volume into face $x = q_x dydz$
$V_x = q_x dydz$

Outflow from x direction $q_x dydz + \frac{\partial q_x}{\partial x} dxdydz$

$$= \left(q_x + \frac{\partial q_x}{\partial_x}\right) dydz \tag{4}$$

4.2 Flow in y- Direction

Volume into face $y = q_y A_y$
where $A_y = dxdz$
volume into face $y = q_y dxdz$
$V_y = q_y dxdz$

Outflow from y direction $q_y dxdz + \frac{\partial q_y}{\partial y} dxdydz$

$$= \left(q_y + \frac{\partial q_y}{\partial_y}\right) dxdz \tag{5}$$

4.3 Flow in z- Direction

Volume into face $z = q_z A_z$
where $A_z = dxdy$
volume into face $z = q_z dxdy$

$$V_z = q_z dxdy$$

Outflow from z direction $q_z dxdy + \frac{\partial q_z}{\partial z} dxdydz$

$$= \left(q_z + \frac{\partial q_z}{\partial_z}\right)dxdy \quad (6)$$

Total volume in flow = $q_x dydz + q_y dxdz + q_z dxdy$ (7)

Total volume outflow = $\left(q_x + \frac{\partial q_x}{\partial_x}\right)dydz + \left(q_y + \frac{\partial q_y}{\partial_y}\right)dxdz + \left(q_z + \frac{\partial q_z}{\partial_z}\right)dxdy$ (8)

Volume in – Volume out = Change in storage

$$q_x dydz + q_y dxdz + q_z dxdy - \left(q_x + \frac{\partial q_x}{\partial_x}\right) dydz - \left(q_y + \frac{\partial q_y}{\partial_y}\right) dxdz - \left(q_z + \frac{\partial q_z}{\partial_z}\right) dxdy$$

$$= -\frac{\partial q_x}{\partial x} dxdydz - \frac{\partial q_y}{\partial y} dxdydz - \frac{\partial q_z}{\partial z} dxdydz$$

$$= \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) dxdydz \quad (9)$$

From continuity of mass, the net accumulation in the control volume where V_w of the volumetric water content and t is the time $= -\frac{\partial V_w}{\partial t}$. The net accumulation control volume is due to movement partial to the x- axis which is equal to the inflow less the outflow.

$$\partial V_w = S_s V_T (-\partial h)$$

but $-\frac{\partial V_w}{\partial t} = S_s V_T \left(\frac{\partial h}{\partial t}\right)$

and $/V_T = \partial x \partial y \partial z$ $/V_T = \partial x \partial y \partial z$

thus $-\frac{\partial V_w}{\partial t} = S_s \frac{\partial h}{\partial t} dxdydz$ (10)

Combining Equation 9 and 10, the net total accumulation of mass is the control volume yields:

$$-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)dxdydz = S_s\frac{\partial h}{\partial t}\,dsdydz - \frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z} = S_s\frac{\partial h}{\partial t} \quad (11)$$

In vector notation $-\nabla \cdot q = S_s\frac{\partial h}{\partial t}$

Applying Darcy's law (for cases where hydraulic conductivity tensor axes are aligned with *x*, *y*, *z* axes):

$$q_x = -K_x\frac{\partial h}{\partial x}$$

$$q_y = -K_y\frac{\partial h}{\partial y}$$

$$q_z = -K_z\frac{\partial h}{\partial z}$$

Where *Kx*, *y*, *z* are directional hydraulic conductivities substituting:

$$\frac{\partial\left(K_x\frac{\partial h}{\partial x}\right)}{\partial x}+\frac{\partial\left(K_y\frac{\partial h}{\partial y}\right)}{\partial y}+\frac{\partial\left(K_z\frac{\partial h}{\partial z}\right)}{\partial z} = S_s\frac{\partial h}{\partial t} \quad (12)$$

Ground water flow equation for porous media which is both heterogeneous and anisotropic. In general, groundwater flow equation. The various simplifying conditions are:

1. Heterogeneous and isotropic

   Hydraulic conductivity $k_x = k_y = k_z$ but $k = k\,(x, y, z)$.

$$\frac{\partial\left(K\frac{\partial h}{\partial x}\right)}{\partial x}+\frac{\partial\left(K\frac{\partial h}{\partial y}\right)}{\partial y}+\frac{\partial\left(K\frac{\partial h}{\partial z}\right)}{\partial z} = S_s\frac{\partial h}{\partial t} \quad (13)$$

2. Anisotropic and homogenous

   Hydraulic conductivity $k_x \neq k_y \neq k_z$ but all *k*'s are constant in space.

$$\frac{\partial\left(K_x\frac{\partial h}{\partial x}\right)}{\partial x} = K_x\frac{\partial\left(\frac{\partial h}{\partial x}\right)}{\partial x} = K_x\frac{\partial^2 h}{\partial x^2}$$

$$\frac{\partial\left(K_y\frac{\partial h}{\partial y}\right)}{\partial y}=K_y\frac{\partial\left(\frac{\partial h}{\partial y}\right)}{\partial y}=K_y\frac{\partial^2 h}{\partial y^2}$$

$$\frac{\partial\left(K_z\frac{\partial h}{\partial z}\right)}{\partial z}=K_z\frac{\partial\left(\frac{\partial h}{\partial z}\right)}{\partial z}=K_z\frac{\partial^2 h}{\partial z^2}$$

Therefore $K_x\frac{\partial^2 h}{\partial x^2}+K_y\frac{\partial^2 h}{\partial y^2}+K_z\frac{\partial^2 h}{\partial z^2}=S_s\frac{\partial h}{\partial t}$ (14)

3. Homogeneous and isotropic

   Hydraulic conductivity $k_x = k_y = k_z$.

$$K\left(\frac{\partial^2 h}{\partial x^2}+K_y\frac{\partial^2 h}{\partial y^2}+K_z\frac{\partial^2 h}{\partial z^2}\right)=S_s\frac{\partial h}{\partial t}$$

$$K\nabla^2 h=S_s\frac{\partial h}{\partial t} \quad (15)$$

4.4 Volumetric Water Content

The volumetric water content in the control volume is equal to $\Phi dxdydz$ where Φ is the porosity. The initial mass of the water is $\rho_w\Phi dxdydz$. The volume of solid material is $1 - \Phi dxdydz$. From Equation 6:

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial z}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial z}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right)=S_s\frac{\partial h}{\partial t} \quad (16)$$

Any change in the mass of water with respect to time is given by:

$$\frac{\partial h}{\partial t}=\frac{\partial}{\partial t}\left(e_w\phi dxdydz\right) \quad (17)$$

As the pressure in the control volume changes, the fluid density will change as the porosity of the aquifer. The compressibility of water (β) is defined as the rate of change in density with a change in pressure (ρ).

$$\beta\rho_w=\frac{d\rho_w}{d\rho}$$

$$\beta d\rho=\frac{d\rho_w}{\rho} \quad (18)$$

The aquifer also changes in volume with a change in pressure. We shall assume the only change is vertical. The aquifer compressibility (α) is given by:

$$adp = \frac{d(dz)}{dz} \tag{19}$$

As the aquifer expands, ϕ will change, but the volume of solid V_s will be constant, likewise, if the only deformation is the *z*- direction, *d*(*dx*) and *d*(*dy*) will be equal to zero.

$$dV_s = d\left[\left(1 - \phi\right)dxdydz\right] \tag{20}$$

Differentiating Equation 19:

$$dzd\phi = \left(1 - \phi\right)d(dz)$$

$$d\phi = \frac{\left(1 - \phi\right)d(dz)}{dz} \tag{21}$$

The pressure (*p*) at a pointing the aquifer is equal to $p_0 + p_w gh$, where p_0 is atmospheric pressure, ρ_w is a constant and *h* is the height of a column of water above the point. Therefore,

$dp = p_w gdh$, Equations 17 and 18 become

$$dp_w = p_w \beta(gdh) \tag{22}$$

and

$$d(dz) = -dz\alpha(p_w gdh) \tag{23}$$

Equation 20 can be rearranged if *d*(*dz*) is replaced by Equation 22,

$$d\phi = \frac{\left(1 - \phi\right)\left(-dz\alpha\rho_w gdh\right)}{dz}$$

$$d\phi = \left(1 - \phi\right)\alpha\rho_w gdh \tag{24}$$

if *dx* and *dy* are constant, the equation for change of mass with time in the control volume, Equation 16 can be expressed as:

$$\frac{\partial h}{\partial t} = \left[\rho_w \phi \frac{\partial(dz)}{\partial t} + \rho_w dz \frac{\partial \phi}{\partial t} + \phi dz \frac{\partial \rho_w}{\partial t}\right]dxdy \tag{25}$$

Substituting Equations 21, 22 and 23 into Equation 24 after minor manipulations yields:

$$\frac{\partial h}{\partial t} = \left[\alpha\rho_w g + \phi\beta\rho_w g\right] dxdydz \frac{\partial h}{\partial t} \tag{26}$$

The net accumulation of volumetric water content expressed in Equation 7 is equal to Equation 25:

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) dxdydz$$

$$\left(\frac{\partial}{\partial x}\rho_w q_x - \frac{\partial}{\partial y}\rho_w q_y - \frac{\partial}{\partial z}\rho_w q_z\right) = dxdydz\left[\alpha\rho_w g + \phi\beta\rho_w g\right] dxdydz \frac{\partial h}{\partial t}$$

$$\left(\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z}\right) = \left(\alpha\rho_w g + \phi\beta\rho_w g\right)\frac{\partial h}{\partial t} \tag{27}$$

From Equations 18, 19, 20 and 22, we obtained:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial z}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial z}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = \left(\alpha\rho_w g + \phi\beta\rho_w g\right)\frac{\partial h}{\partial t} \tag{28}$$

For isotopic medium, we have,

$$k\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right) = \left(\alpha\rho_w g + \phi\beta\rho_w g\right)\frac{\partial h}{\partial t} \tag{29}$$

Equation 29 is the main equation of flow for a confined aquifer which is the general equation for groundwater flow in three dimensional, we introduce storativity (*S*):

$$S = m\left(\alpha\rho_w g + \phi\beta\rho_w g\right) \tag{30}$$

and transmissivity (T) = km where m is the aquifer thickness.

Therefore, from Equations 29 and 10, we have:

$$k\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right) = \frac{S}{m}\frac{\partial h}{\partial t} = \frac{S}{T/K}\frac{\partial h}{\partial t} = \frac{S}{T}\cdot\frac{\partial h}{\partial t}$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S}{T}\cdot\left(\frac{K^1}{\mathrm{K}}\right)\frac{\partial h}{\partial t} = \frac{S}{T}\frac{\partial h}{\partial t} \tag{31}$$

$\frac{K^1}{\mathrm{K}} \longrightarrow 1$ and where $\mathrm{S_s} = \frac{S}{T}$ for a confined aquifer.

A complete formulation of a generalised groundwater flow equation is obtained by combining mass conservation equation, Darcy's law and definition of specific storage equation, which require specifying:

1. the extent of the flow domain by assumptions,
2. the governing equation,
3. spatial distribution of properties e.g. hydraulic conductivity (k) and hydraulic head (h), specific storage (s_s),
4. boundary conditions and
5. initial conditions.

In general, a generalised equation of groundwater flow equation in three-dimensional equation is expressed as:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial z}\right)+\frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial z}\right)+\frac{\partial}{\partial z}\left(k_z \frac{\partial h}{\partial z}\right)=S_s \frac{\partial h}{\partial t} \tag{32}$$

Under this condition, instead of considering the flow in three dimension with $h(x.y.t)$ since the flow is two-dimensional in the horizontal xy plane, all terms involving first and second derivative with respect to z- axis components vanish for flow in an aquifer that is plane surface.

The two-dimensional, transient two of groundwater is a confined isotropic aquifer is governed by the partial differential equation.

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial z}\right)+\frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial z}\right)=S_s \frac{\partial h}{\partial t} \tag{33}$$

The one-dimensional, transient flow of groundwater in a confined isotropic aquifer is governed by the partial differential equation. For 1-D problem, the pumping well is fully penetrating a non-leaking aquifer and the hydraulic conductivity (k) is an isotropic scaler. The assumption here is that the flow in an aquifer is essentially in horizontal direction and this assumed independent of y-and z-axis.

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial z}\right)=S_s \frac{\partial h}{\partial t}$$

$$\frac{\partial^2 h}{\partial x^2}=\frac{S_s}{K}\frac{\partial h}{\partial t}$$

$$\nabla^2 h=\frac{S_s}{K}\frac{\partial h}{\partial t} \tag{34}$$

where,

K = hydraulic conductivity,
h = hydraulic head,
t = the time and
s_s = specific storage.

This equation is valid of a continuous aquifer, for which there is a value of K, t, S_s and h everywhere.

4.5 Equation of Unsteady Flow in a Leaky Aquifer

Another useful coordinate system is three dimensional cylindrical coordinates (typically where a pumping well is a line source located at the origin parallel to the z- axis causing converging radial flow). Under these conditions, the above Equation 34 becomes (r being radial distance):

$$T\left[\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}\right]-S\frac{\partial}{\partial t}-q=0 \quad (35)$$

We account for leakage using:

$$q=\frac{k^i}{b^i}S=\frac{s}{c^i} \quad (36)$$

Dividing by the aquifer transmissivity yields the equation:

$$\frac{\partial^2 S}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}-\frac{1}{D}\frac{\partial s}{\partial t}-\frac{S}{L^2}=0$$

$$\text{where } L^2=TC \quad (37)$$

Using the same approach as the solution for the confined (Theis) solution, we obtain the leaky partial differential equation:

$$n\left[\frac{\partial^2 S}{\partial u^2}+\frac{\partial s}{\partial u}\right]-\frac{\partial S}{\partial t}-\frac{r^2}{4uL^2}S=0$$

This is equivalent to:

$$n^2\frac{\partial^2 S}{\partial u^2}+u(u+1)-\frac{r^2}{4L^2}S=0 \quad (38)$$

We can find an approximate solution for Jacob condition i.e. $n < 1$.

$$n^2 \frac{\partial^2 S}{\partial u^2} + u\frac{\partial s}{\partial u} - (u^2 + v^2)S = 0 \tag{39}$$

From Equations 38 and 39, we have:

$$u^2 + v^2 = \frac{r^2}{4L^2} \tag{40}$$

$$\text{where } V = \sqrt{\frac{r^2}{4L^2} - u^2} \tag{41}$$

yielding,

$$S = \frac{Q}{4\pi T} W\left(u, \frac{r}{B}\right) \tag{42}$$

The quantity $\frac{r}{B}$ is given by $\frac{r}{B} = \frac{r}{\sqrt{T/\left(k^i/b^i\right)}}$ which holds as long as u < 0.01 and $\frac{r}{2L} > u$. Hantush and Jacob found an approximate solution to the leaky equation by setting: $S = \frac{\partial s}{\partial u}$ to obtain $u\frac{\partial^2 s}{\partial u^2} + \left(u + 1 - \frac{r^2}{4uL^2}\right)\frac{\partial s}{\partial u} = 0$, multiplying through by two integrating factors yields, $e^u e^{\frac{r^2}{4uL^2}}\left[u\frac{\partial^2 s}{\partial u^2} + \left(u + 1 - \frac{r^2}{4uL^2}\right)\frac{\partial s}{\partial u}\right] = 0$ which is equivalent to $\frac{\partial\left[ue^u e^{\frac{r^2}{4uL^2}}\right]}{\partial u} = 0$ so that,

$$S^i = C^{\frac{e^{-u-\frac{r^2}{4uL^2}}}{u}} = 0 \tag{43}$$

Again applying the borehole wall condition, $Q = 2\pi rT\frac{\partial s}{\partial r} = 4\pi rT\frac{\partial s}{\partial u}$, we obtain:

$$\frac{\partial s}{\partial u} = \frac{Q}{4\pi T}\frac{e^{-u-\frac{r^2}{4uL^2}}}{u} = 0$$

$$S = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-u-\frac{r^2}{4uL^2}}}{x}dx \tag{44}$$

which is Hantush–Jacob solution.

4.6 Solution to the Problem

Suppose that the three edges $x = 0$, $x = L$ and $y = 0$ of a thin rectangular plate are maintained at zero temperature, h(0,y) = h(l, y) = h(x, 0) = 0 (45) and that the fourth edge $y = h$ is maintained at a hydraulic distribution $f(x)$,

$$h(x, H) = f(x) \tag{46}$$

until steady state conditions are realised as shown in Figure 3.

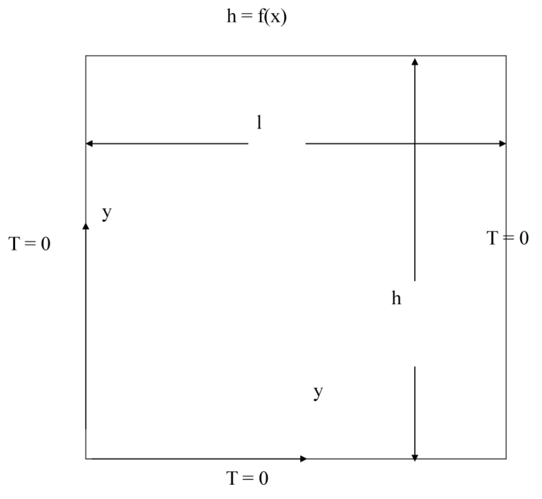

Figure 3: An illustration of steady state condition.

The hydraulic distribution throughout the plate is required. Thus we must determine that solution of Laplace's equation in two dimensions, $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$ (47) which takes on the prescribed boundary values in Equations 45 and 46. The method of separation of variables consists of seeking particular "product solutions" of equation 47 in the form.

$$h_v(x, y) = x\,(u),\, y(y), \tag{48}$$

where X is a function of x alone and Y is a function of y alone. Introducing Equation 48 into Equation 47, there follows $\frac{\partial^2 x}{\partial x^2} Y + X \frac{\partial^2 y}{\partial y^2} = 0$ or, separating the variables,

$$-\frac{1}{X}\frac{\partial^2 x}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 y}{\partial y^2} \tag{49}$$

Since by hypothesis, the left-hand member of Equation 49 is independent of y and the equivalent right-hand member is independent of x, it follows that both sides must be independent of both x and y, and hence must be equal to a constant. If we call this arbitrary constant K^2, there follows:

$$\frac{\partial^2 x}{\partial x^2} + K^2 X = 0 \tag{50[a]}$$

$$\frac{\partial^2 y}{\partial y^2} + K^2 Y = 0 \tag{50[b]}$$

Thus we see that the product (Equation 48) will satisfy (Equation 47) if x and y are solutions of Equation 50, regardless of the value of k. Because of the linearity of Equation 47, it follows that any linear combination of such solutions, corresponding to different values of k, will also satisfy Equation 47.

It is noticed that three of the boundary conditions are homogeneous. Thus, if each of the particular product solutions is required to satisfy Equation 45, any linear combination will also satisfy the same conditions. Equation 45 will be satisfied if:

$$x(0) = x(L) = 0 \tag{51}$$

whereas k implies the condition:

$$y(0) = 0 \tag{52}$$

Equations 50a and 50b constitute a previously considered Sturm-Lioville problem for which the characteristic values are:

$$k = k_n = \frac{n\pi}{L}\left(n = 1,2,3,\ldots\right), \tag{53}$$

and the corresponding solutions (characteristic functions) are of the term:

$$X = X_n = A_n \sin\frac{n\pi x}{L} \tag{54}$$

Corresponding to (Equation 52), the solution of (Equation 50b) which satisfies (Equation 52) is of the form.

$$Y = Y_n = B_n \sinh\frac{n\pi y}{L} \tag{55}$$

Thus it follows that any particular solution of the form:

$$h_n = a_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}\left(n = 1,2,3,\ldots\right) \tag{56}$$

where we have written $a_n = A_n B_n$, satisfies equation (Equation 47) and the three boundary conditions (Equation 45) which is true for any series of the form.

$$h = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L} \tag{57}$$

if suitable convergence is assumed. It remains, then, to attempt to determine the coefficients a_n in Equation 57 in such a way that the remaining condition Equation 46 is satisfied, so that:

$$f(x) = \sum_{n=1}^{\infty}\left(a_n \sinh\frac{n\pi H}{L}\right)\sin\frac{n\pi x}{L}(0 < x < L) \tag{58[a]}$$

But, from the theory of Fourier sine series, the coefficient $a_n \sinh\frac{n\pi H}{L}$ in this series must be of the form: $a_n \sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$ and hence, writing $C_n = a_n \sinh(n\pi H/L)$, the required solution Equation 57 takes the form,

$$h(x,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\frac{\sinh\frac{n\pi y}{L}}{\sinh\frac{n\pi H}{L}} \tag{58[b]}$$

$$\text{where } C_n = \frac{2}{L}\int_{\infty}^{L} f(x)\sin\frac{n\pi x}{L}dx \tag{59}$$

assuming appropriate convergence. Let the hydraulic head of five faces of a rectangular parallel piped be maintained at zero,

$$H(0, y, z) = h(L, y, z) = h(x, 0, z) = h(x, 0, z) = h(x, L_2, z) = h(x, y, 0) = 0 \tag{60}$$

and suppose that the sixth face is maintained at a prescribed hydraulic distribution,

$$h(x, y, H) = f(x, y) \tag{61}$$

until steady-state conditions are attained. We again investigate the resultant distribution of hydraulised in the interior. If we assume a product reduction of the relevant equation,

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \tag{62}$$

in the form:

$$h_v = x(u)Y(y)Z(z). \tag{63}$$

The equation may be separated in the form:

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z} = K_1^2 \tag{64}$$

The separation here depending upon the fact that the first member is independent of both y and z and the second equal member is independent of x. Hence we must have:

$$X'' + K_1^2 X = 0 \tag{65}$$

and after a second separation,

$$\frac{Y''}{Y} = \frac{Z''}{Z} - K_1^2 = K_2^2 \tag{66}$$

Thus Y and Z are determined by the equations:

$$Y'' + K_2^2 Y = 0 \tag{67}$$

$$Z'' - \left(K_1^2 + K_2^2\right)Z = 0 \tag{68}$$

The homogeneous boundary conditions Equation 60 are satisfied by the product solution if the factors satisfy the conditions.

$$X(0) = X(L_1) = 0 \tag{69[a]}$$

$$Y(0) = Y(L_2) = 0 \tag{69[b]}$$

$$Z(0) = 0 \tag{69[c]}$$

We thus obtain from Equations 65, 67, 69(a) and 69(b):

$$K_1 = \frac{m\pi}{L_1}\left(m = 1,2,3\ldots\right) \tag{70}$$

$$X = X_m = A_{\mathrm{m}} \sin\frac{m\pi x}{L_1} \tag{71}$$

$$K_2 = \frac{n\pi}{L_2}\left(n = 1,2,3\ldots\right) \tag{72}$$

$$Y = Y_n = B_n \sin\frac{n\pi y}{L_2} \tag{73}$$

To write further $K_2^1 + K_2^2 = \pi^2\left(\frac{m^2}{L_1^2} + \frac{n^2}{L_2^2}\right) = K_{mn}^2$

$$Or K_{mn} = {}^{\pi}\sqrt{\frac{m^2}{L_1^2} + \frac{n^2}{L_2^2}} \tag{74}$$

The solution of Equation 68 satisfying Equation 69(c) becomes:

$$Z_{mn} = C_{mn} \sinh K_{mn} Z \tag{75}$$

Thus, writing $a_{mn} = A_m B_n C_{mn}$, we are led to assume the desired solution in the form:

$$h(x,y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} \sinh K_{mn} Z \tag{76}$$

This expression satisfies Equation 19, as well as conditions of Equation 17 for arbitrary values of the coefficients a_{mn}. It remains, then, to determine these coefficients, in such a way that the remaining condition of Equation 18 is satisfied if we introduce the abbreviation:

$$\mathrm{C_{mn} = a_{mn} \sinh K_{mn} H} \tag{77}$$

This condition takes the form of:

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} \left(0 < x < h, 0 < y < L_2\right) \tag{78}$$

Thus the coefficients C_{mn} are the coefficients of the double Fourier sine-series expansion of $f(x,y)$ over the indicated rectangle (Figure 4).

These coefficients are readily determined by a simple extension of the methods used in earlier work. If both Equations 77 and 78 are multiplied by $\sin\left(\frac{p\pi x}{L_1}\right)\sin\left(\frac{q\pi y}{L_2}\right)$ where p and q are arbitrary positive integers and if the results are integrated over the rectangle, there follows:

$$\begin{aligned} &\int_0^{L_1}\int_0^{L_2} f(x,y) \sin\frac{p\pi x}{L_1} \sin\frac{q\pi y}{L_2} dxdy \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \int_0^{L_1}\int_0^{L_2} \sin\frac{p\pi x}{L_1} \sin\frac{q\pi y}{L_2} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} dxdy \end{aligned} \tag{79}$$

The double integral on the right can be written as the product:

$$\left[\int_0^{L_1} \sin\frac{p\pi x}{L_1}\sin\frac{m\pi x}{L_1}\right]\left[\int_0^{L_2} \sin\frac{q\pi y}{L_2}\sin\frac{n\pi y}{L_2}\right] \quad (80)$$

and hence, this product vanishes unless $p = m$ and $q = n$, in which case it has the value: $\frac{L_1}{2}\frac{L_2}{2} = \frac{L_1 L_2}{4}$. Thus the double series in the right-hand member of Equation 79 reduces to a single term, for which $m = p$ and $n = q$ and there follows:

$$C_{mn} = \frac{4}{L_1 L_2}\int_0^{L_1}\int_0^{L_2} f(x,y)\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}dxdy \quad (81)$$

with these values of C_{mn}, the solution of Equation 76 becomes, with the notation of Equation 77:

$$h(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\frac{\sinh K_{mn}Z}{\sinh K_{mn}} \quad (82)$$

where K_{mn} is defined by Equation 74.

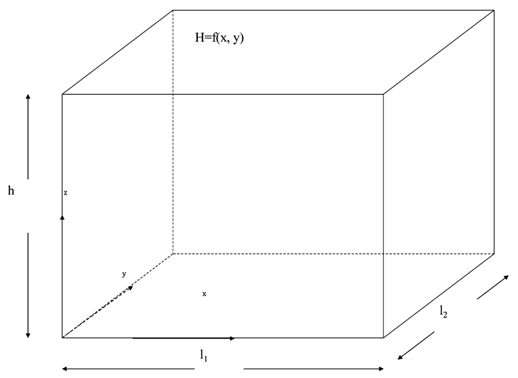

Figure 4: A 3-D illustration.

4.7 Pressure and Permeability

Buoyancy forces due to temperature changes can be a driving mechanism for fluid flow. Because a geothermal gradient occurs in the subsurface, fluids at depth are hotter than fluids near the surface. The hotter fluids are less dense and lighter than the cooler fluids and tend to rise conversely, the cooler fluids are denser and heavier than the hotter fluids and tend to sink. Under certain condition, buoyancy-driven flow can occur in the absence of topographic driven flow is known as thermal convection.

Recall that Darcy's law in terms of hydraulic head h is written as:

$$q = -K\nabla h \tag{83}$$

Where q is the specific discharge vector and k is hydraulic conductivity. When changes in the fluid density are important, it is necessary to express Darcy's law in terms of pressure, p. This form of Darcy's law can be written as:

$$q = -K\nabla\left[\frac{p}{\rho_w g} + z\right] \tag{84}$$

where ρ_w is the fluid density, g is acceleration due to gravity, and z is the elevation (above a given datum) of the point at which hydraulic head is measured because hydraulic conductivity is related to permeability (k) by:

$$k = \frac{\rho_w g k}{\mu} \tag{85}$$

where u is the dynamic viscosity of the fluid, Equation 46 can be written as:

$$q = \frac{\rho_w k}{\mu}\nabla\left[\frac{p}{\rho_w} + gz\right] \tag{86}$$

For a slightly compressible fluid, it is generally acceptable to assume that the spatial variation in fluid density is small so that Equation 46 can be written as:

$$q = -\frac{k}{\mu}\nabla\left(p + \rho_w gz\right) \tag{87}$$

Equation 50 is the common form of Darcy's law expressed in terms of pressure and permeability. The vector components of this equation are:

$$q_x = -\frac{k}{\mu}\frac{\partial p}{\partial x} \tag{88}$$

$$q_y = -\frac{k}{\mu}\frac{\partial p}{\partial y} \tag{89}$$

$$q_z = -\frac{k}{\mu}\left[\frac{\partial p}{\partial x} + \rho_w g\right] \tag{90}$$

where Equation 90 is the groundwater flow in geological formation.

5. DISCUSSION OF RESULTS

Flow in porous media has always been a matter of great interest and of great importance to Mathematicians. Certain geological formations are typical of porous media comprising matrix of particles. Theoretically, when the void spaces in such media are only partially filled with water, the water is usually attracted to the particle surfaces through electrostatic forces between the water molecules' polar bonds and the particle substances. The surface attention draws the water up around the particle surfaces, leaving the air in the centre of the voids. As more water is added to the porous medium via the hydrological cycle, the air exists upwards and the area of free surface diminishes within the medium until the medium is saturated and there are no free surfaces within the voids and therefore, no soil suction force.

In order to advance the study of flow through porous media, hydrogeological data are being analysed. This would further contribute to the unveiling knowledge of the applicability of flow in porous media. Table 1 is a result of pumping test conducted on a 200 mm well at the rate 1,150 1pm. The observation well 12.3 m away from the pumped well. Mathematical modelling approach makes it possible to determine the transmissibility and storage coefficients of the aquifer. Furthermore, what the drawdown would be at the end of 180 days both in the observation well, and in the pumped well can be computed using modified Theis equation.

Table 1: Time and drawdown data of a water well pumping test (adapted).[20]

Time (min)	2	3	5	7	9	12	15	20	40	60	90	120
Drawdown (m)	2.42	2.46	2.52	2.58	2.61	2.63	2.67	2.71	2.79	2.85	2.91	2.94

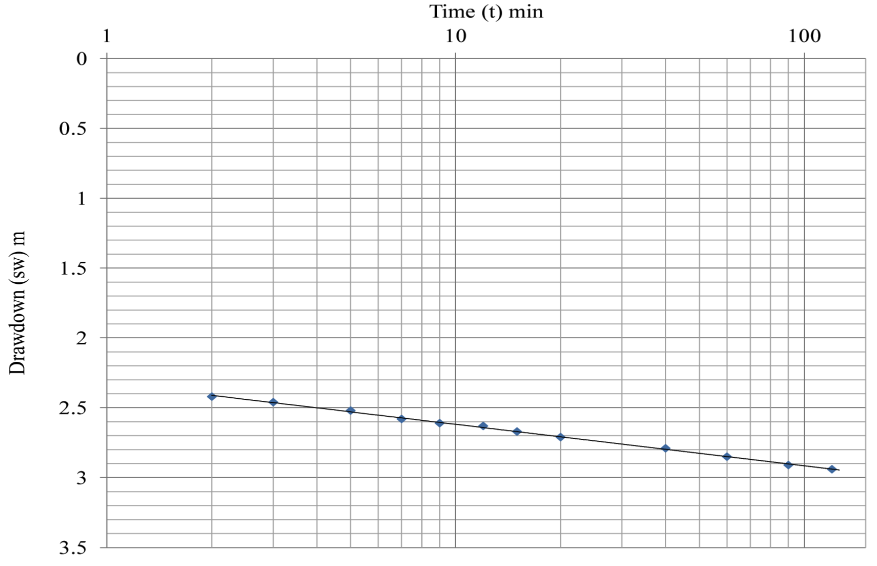

Figure 5: Drawdown-time semi-log plot.

The time-drawdown plot is shown in Figure 5, from which $\Delta s = 0.28$ m per log-cycle of t, t_0 (for $s_w = 0$) is 37×10^{-10} min.

$$\therefore T = \frac{2.3Q}{4\pi\Delta s} = \frac{2.3 \times 1.150/60}{4\pi(0.28)} = 0.0125 m^2/s$$

$$s_w = \frac{2.25T_{t_0}}{r^2} = \frac{2.25(0.0125)37 \times 10^{-10} \times 60}{(12.3)^2} = 4.12 \times 10^{-11}$$

1. Drawdown in the observation well after 180 days:

$$s_w = \frac{2.3Q}{4\pi T}\log\frac{2.23Tt}{r^2 s}, \qquad u < 0.01$$

$$s_w = \frac{2.31.150/60}{4\pi(0.0125)}\log\frac{2.23(0.0125)180 \times 86400}{(12.3)^2 4.12 \times 10^{-11}}$$

$$= 3.89m$$

2. Drawdown in the pumped well after 180 days:

$$s_w = \frac{2.3(1.150/60)}{4\pi(0.0125)}\log\frac{2.25(0.0125)180 \times 86400}{(0.100)^2 4.2 \times 10^{-11}}$$

$$= 3.89m$$

The Jacob's method is valid for:

$$u < 0.01$$

$$\frac{r^2 S}{4Tt} < 0.01$$

or

$$t > \frac{r^2 S}{0.04T}$$

$$> \frac{(12.3)^2 4.12 \times 10^{-11}}{0.04(0.0125)}$$

$$> 1.25 \times 10^{-5}$$

i.e. instateneously after pumping starts

Another production well is pumped for two hours at a constant rate of 1,600 1pm. The drawdown in seven observation wells is shown in Table 2 below. In this case it is possible to determine the aquifer storage coefficient (*S*) and transmissibility (*T*).

Table 2: Observation well distance and drawdown from pumping test data (adapted).[20]

Observation well	A	B	C	D	E	F	G
Distance from pumped well (m)	5	10	20	40	80	120	200
Drawdown (m)	5.35	4.35	3.35	2.35	1.4	0.8	0.3

The distance-drawdown plot is shown in Figure 6 from which $\Delta s = 3.25$ m per log-cycle of r, r_0 (for $s = 0$) is 210 m.

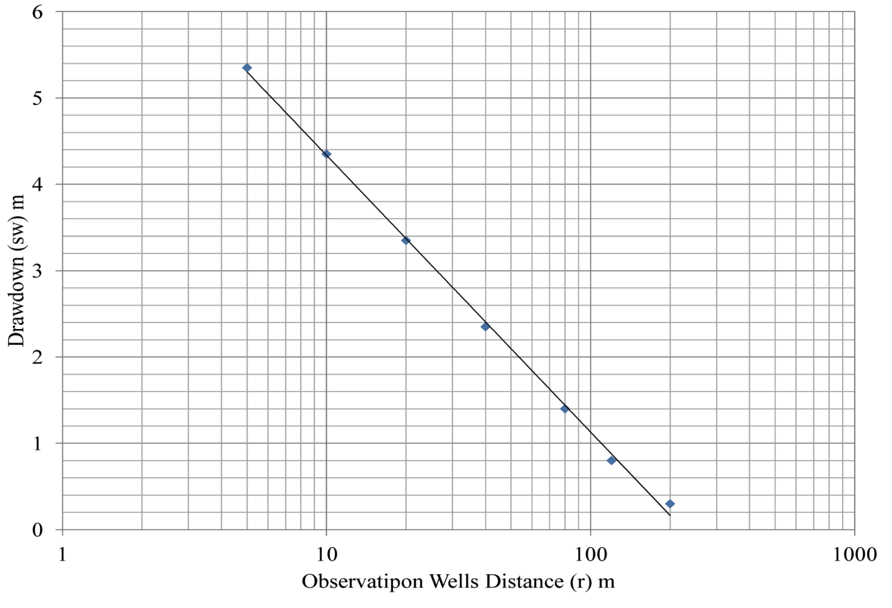

Figure 6: Drawdown-observation wells distance semi-log curve.

$$T = \frac{2.3Q}{2\pi\Delta s} = \frac{2.3(1.150/60)}{2\pi(3.25)} = 0.03\frac{m^2}{s} \text{ or } 2.6\times10^6\ 1\,pd/m$$

$$S = \frac{2.25Tt_0}{r_0^2} = \frac{2.25(0.03)2\times60\times60}{210^2} = 0.011$$

Finally, a 400 mm well pumped at the rate of 2,000 1pm for 200 min yielded a drawdown of 1.51 m in an observation well 20 m from the pumping well. The pumping was stopped and the residual drawdowns during recovery in the observation well for two hours are given in Table 3. In a similar way, it is convenient to determine the aquifer storage and transmissibility.

The time-residual drawdown data are processed in Table 3 and the Theis recovery curve is plotted on a semi-log paper as shown in Figures 7.

Table 3: Pumping test conducted on water well for Theis solution t_1 = 200 min.

Time since pumping stopped t' (min)	Residual drawdown s' (m)	Time since pumping started $t = t_1 + t'$ (min)	Ratio t/t'
2	0.826	101	202
3	0.664	68	203
5	0.549	41	205
10	0.427	21	210
16	0.351	13.5	216
20	0.305	11	220
25	0.271	9	225
30	0.241	7.7	230
35	0.220	6.7	235
40	0.201	6	240
45	0.180	5.45	245
50	0.159	5	250
55	0.155	4.65	255
60	0.149	4.33	260
70	0.146	3.86	270
80	0.140	3.5	280
90	0.134	3.22	290
100	0.131	3	300
110	0.131	2.82	310
120	0.131	2.66	320

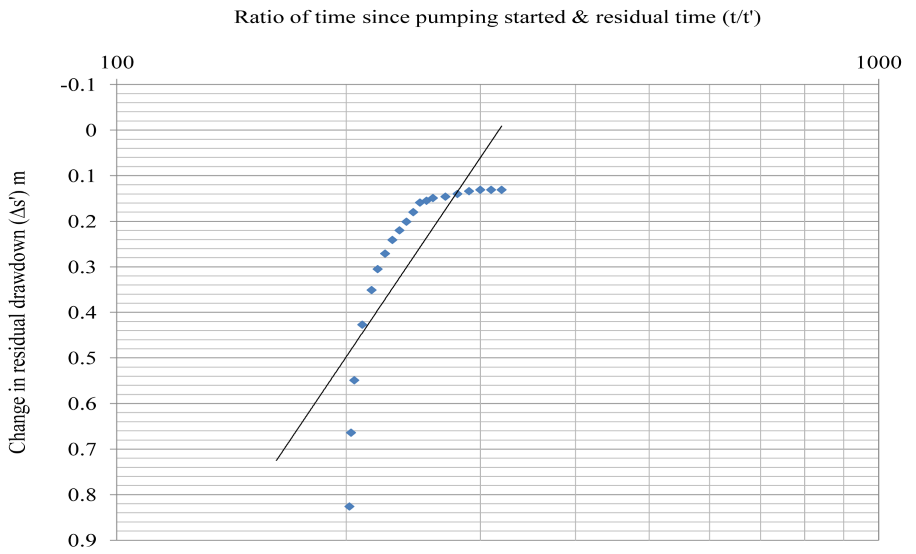

Figure 7: Drawdown-time semi-log plot.

From the recovery plot, Δ*s*' = 0.41 m per log-cycle of *t*/*t*' and $T = \frac{2.3Q}{4\pi\Delta s'} = \frac{2.3(2.000/60)}{4\pi(0.41)} = 0.0149 m^2/s$ or $= 1.284 \times 10^6\ 1\,pd/m$, and *S* can be obtained from s_1= 1.51 m after 200 min of pumping as:

$$s_1 = \frac{2.3Q}{4\pi T}\log\frac{2.25Tt_1}{r^2 s}$$

$$\log\frac{2.25Tt_1}{r^2 s} = \frac{4\pi(0.0149)1.51}{2.3(2.000/60)} = 3.69$$

Antilog of 3.69 = 4898

$$\frac{2.25(0.0149)200 \times 60}{20^2 S} = 4898$$

$$\therefore S = 0.000206$$

6. SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

This study relates water flowing in porous media to hydraulic conductivity and permeability taking cognisance of the complexity of a geological formation. Darcy law is reduced to the form of Lapace equation in two-dimensional form with series of curves of drawdown-time, and drawdown-observation wells distance. Time of pumping increases with an increase in the value of the drawdown. Both the analytical and numerical methods using finite difference show that water flow from a higher gradient to a gradient concentration. Drawdown increases as the observation well distance increases. Further researches should be encouraged to undertake findings in multiphase flow through a porous media. A comprehensive study is recommended on surfaced water and groundwater flow direction relationship. Study on mechanics of groundwater flow in porous media be extended to physical behaviour of fluids flow and their interaction with solid matrix.

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