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Analytical Solution for the Boundary Value Problem of Euler-Bernoulli Beam Subjected to Accelerated Distributed Load

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Abstract: *The study of dynamic response of beam-like structures to moving or static loads has attracted and still attracting a lot of attention due to its wide range of applications in the construction and transportation industry especially when transverse by travelling masses. Hence, analytical solution for the boundary value problem (BVP) of elastic beams subjected to distributed load was investigated. The partial differential equation of order four were analysed to determine the dynamic response of the elastic beam under consideration and solved analytically. Effects of different parameters such as the mass of the load, the length of the moving load, the distance covered by the moving load, the speed of the moving and the axial force were considered. Result revealed that the values of the deflection with acceleration being considered increases than the system where acceleration of the moving load is negligible.*

Keywords: Euler-Bernoulli beam, axial force, accelerated distributed load, dynamic response

1. INTRODUCTION

Most physical and engineering boundary value problem (BVP) can be modelled as functional equations. However, for most of these equations, exact solutions are very rare. Several analytical and numerical methods are being developed to obtain approximate solutions for such models.[1] Analytical solutions for BVPs are always preferable compared to numerical solutions as they are more general and give a better understanding of the model behaviour. Due to great practical importance for technical engineers, the analysis of the dynamic behaviour of beams on elastic foundations has attracted many researchers in the century and has therefore been the object of study by a huge number of researchers until now. Many researchers have studied the dynamic structure of a distributed load subjected to various types of load.[2–10] They have formulated the problem using the analytical and numerical techniques since the pioneer studies of Winkler.[11]

The moving load problem is a foundation problem in structural dynamics. Engineers have been investing the potential hazard produced by the moving masses on structures for several years. The dynamic response of structures carrying moving mass is a problem of wide spread practically significant.[12]

Oniszczuk[12] examined the exact theoretical solutions of undamped vibrations for a simply supported Euler-Bernoulli beam. Gbadeyan[13] studied the problem of the dynamic response of beams connected viscoelastically and subjected to uniform partially distributed moving force. Result indicated that there was an increase in the amplitude of the beam for various values of the speed of the moving force considered, such that, as the speed of the moving force increases, the amplitude of the beam also increases. Furthermore, Usman and Hammed[14] considered the dynamic response of a Bernoulli-Beam with viscous and structural damping coefficients subjected to partially distributed moving load. It was observed that as the mass of the beam, increases, the deflection also increases. However, the amplitude of the deflection increases when there are no structural and damping coefficients.

From the mentioned literature, studies have only considered the moving load problem at constant speed, which can be traced to simplicity in solving compared to when acceleration is considered. However, the study of the effect of acceleration on the moving load problem is very scanty. Esen[15] concluded that acceleration of a travelling mass over a structural system highly affects the dynamic response of the structural system. But it is necessary to put into consideration, the acceleration of the moving load.

As a matter of fact, the problem of moving loads is dated back to the beginning of the 19th century,[16] the time of erection of the early railway bridges. In studying the strength of bridges, the problem of determining the stresses and deflections of a beam and a plate under the action of a moving load presented itself.

The problem of moving load was first tackled for the case in which structure was considered small against the mass of a single, constant load. Abdul[17] considered the dynamic behaviour of an elastic beam on a Winkler foundation under a moving load. He assumed the mass of the beam to be smaller than the mass of the load (*M*) and obtain an approximate solution to the problem.

Although the above completed works on concentrated loads are impressive, they do not represent the physical reality of the problem formulations as concentrated masses do not exist physically. Thus, for practical applications, it is realistic to consider the moving load problems as partially distributed moving loads as opposed to concentrated moving loads. To this end, this article investigates the analytical solution for the BVP for Euler-Bernoulli beam subjected to accelerated distributed moving load.

2. MATERIALS AND METHODS

The system to be investigated is that of a BVP under the distributed load resting on a Winkler foundation is considered (as shown in Figure 1). The beam is assured to be length (*L*) on a Winkler foundation.

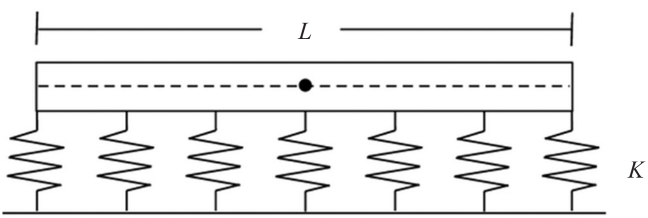

Figure 1: Representation of a beam on a Winkler foundation.

In this article, the following assumptions are used in the analysis of the beam on Winkler foundation under distributed moving loads:

1. The behaviour of the beam can be described by Euler-Bernoulli theorem.
2. The beam is of constant cross-section and constant per unit length.
3. The speed of the distributed moving load is constant.
4. The acceleration of the moving load varies with time.
5. The beam mounted on a Winkler foundation and subjected to a moving load.

The model for the problem under consideration is relatively simple Winkler model. According to theory of structural vibration on using Euler-Bernoulli beam,[17] the governing equation is formulated as:

$$EI\frac{\partial^4 W(x,t)}{\partial x^4}+m\frac{\partial^2 W(x,t)}{\partial x^2}-\rho_0\frac{\partial^2 W(x,t)}{\partial x^2}+KW(x,t)=f(x,t) \quad (1)$$

where

$$f(x,t)=\frac{1}{E}\left[-Mg-M\frac{\partial^2 W}{\partial x^2}\right]B \quad (2)$$

$$B=\delta(x-\upsilon t)=\frac{dH(x)}{dt^2} \quad (3)$$

and K is the Winkler foundation.

In Equation 1, the one parameter, K, was used to model the elastic support based on classical Winkler model.

2.1 Classical Boundary Conditions

The initial condition of the beam at rest is given as:

$$\left.\begin{aligned} W(x,0)&=0\\ \frac{dW(x,0)}{dt}&=0\end{aligned}\right\} \quad (4)$$

and the pertinent boundary conditions for the problem under consideration are:

$$\left.\begin{aligned} W(x,t)=\frac{\partial W(x,t)}{\partial x}&=0 \text{ at } x=0 \text{ or } x=L\\ W(x,t)=\left.\frac{\partial^2 W(x,t)}{\partial x^2}\right|_{x=0}&=0 \text{ at } x=0 \text{ or } x=L\\ \frac{\partial^2 W(x,t)}{\partial x^2}=\frac{\partial^3 W(x,t)}{\partial x^3}&=0 \text{ at } x=0 \text{ or } x=L\\ \frac{\partial W(x,t)}{\partial x}=\frac{\partial^3 W(x,t)}{\partial x^3}&=0 \text{ at } x=0 \text{ or } x=L\end{aligned}\right\} \quad (5)$$

The external load in Equation 1 is defined by:

$$f(x,t) = \frac{1}{E}\left[-Mg - M\frac{\partial^2 W}{\partial x^2}\right]B \tag{2}$$

where

$$B = \delta(x-\upsilon)t = \frac{dH(x)}{dt^2} \tag{3}$$

The reaction force (*f*) exerted by mass (*m*) is defined as:

$$\begin{aligned}\frac{\partial^2 W(x_f,t)}{\partial t^2} &= \frac{\partial^2 W(x,t)}{\partial t^2} + 2V\frac{\partial^2 W(x,t)}{\partial t \partial x} + V^2\frac{\partial^2 W(x,t)}{\partial x^2} \\ &= \frac{\partial^2 W(x_f,t)}{\partial t^2} + 2\left(\frac{\partial x_f(x,t)}{\partial t}\right)\frac{\partial^2 W(x,t)}{\partial t \partial x} + \left(\frac{\partial x_f}{\partial t}\right)^2\frac{\partial^2 W(x,t)}{\partial x^2}\end{aligned} \tag{6}$$

where

$$x_f(t) = x_0 + \ddot{x}_0 t + \frac{a_n t^2}{2} \tag{7}$$

$$\frac{dx_f(t)}{dt} = \ddot{x} + a_n t \tag{8}$$

$$\frac{d^2 x_f(t)}{dt^2} = a_n \tag{9}$$

Substituting Equations 6, 7 and 8 into Equation 1:

$$\begin{aligned}&EI\frac{\partial^4 W(x,t)}{\partial x^4} + m\frac{\partial^2 W(x,t)}{\partial t^2} - \rho_0\frac{\partial^2 W(x,t)}{\partial x^2} + KW(x,t) \\ &= \left(\frac{1}{E}\right)\left(-Mg - M\frac{\partial^2 \psi}{\partial x^2}\right)B\end{aligned} \tag{10}$$

$$\begin{aligned}&EI\frac{\partial^4 W(x,t)}{\partial x^4} + m\frac{\partial^2 W(x,t)}{\partial t^2} - \rho_0\frac{\partial^2 W(x,t)}{\partial x^2} + KW(x,t) \\ &= \left(\frac{1}{E}\right)\left[-M\left(g - \left(\frac{\partial^2 W(x,t)}{\partial t^2} + 2(\ddot{x} + a_n t)\frac{\partial^2 W(x,t)}{\partial t \partial x} + a_n\frac{\partial^2 W(x,t)}{\partial x^2}\right)\right)\right]B\end{aligned} \tag{11}$$

2.2 Solution Technique

In this section, we proceed to solve the initial value problem described by Equation 11. Assume a solution of the form:

$$W(x,t) = \sum_{j=1}^{\infty} q_j(t)\sin\frac{j\pi x}{L} \tag{12}$$

Differentiating Equation 12 with respect to x and with respect to t,

$$\frac{\partial^4 W(x,t)}{\partial x^4} = \left(\frac{j\pi}{L}\right)^4 \sum_{j=1}^{\infty} j^4 q_j(t)\sin\frac{j\pi x}{L} \tag{13}$$

Also,

$$\frac{\partial^2 W(x,t)}{\partial t^2} = \sum_{j=1}^{\infty} \ddot{q}_j(t)\sin\frac{j\pi x}{L} \tag{14}$$

Next,

$$\frac{\partial^2 W(x,t)}{\partial t^2} = \left(\frac{j\pi}{L}\right)^2 \sum_{j=1}^{\infty} j^2 q_j(t)\sin\frac{j\pi x}{L} \tag{15}$$

Substituting Equations 12, 13, 14 and 15 into Equation 11, we have:

$$\begin{aligned}
&EI\left(\frac{j\pi}{L}\right)^4 \sum_{j=1}^{\infty} j^4 q_j(t)\sin\frac{j\pi x}{L} + m\sum_{j=1}^{\infty} \ddot{q}_j(t)\sin\frac{j\pi x}{L} \\
&-\rho_0\left(\frac{j\pi}{L}\right)^2 \sum_{j=1}^{\infty} j^2 q_j(t)\sin\frac{j\pi x}{L} + K\sum_{j=1}^{\infty} q_j(t)\sin\frac{j\pi x}{L} \\
&= \left(\frac{1}{\epsilon}\right)\Bigg[-M\Bigg(g - \Bigg(\sum_{j=1}^{\infty} \ddot{q}_j(t)\sin\frac{j\pi x}{L} \\
&+2(\ddot{x} + a_n t)\left(\frac{j\pi}{L}\right)\sum_{j=1}^{\infty} \dot{q}_j(t)\cos\frac{j\pi x}{L} \\
&+a_n\left(\frac{j\pi}{L}\right)^2 \sum_{j=1}^{\infty} j^2 q_j(t)\sin\frac{j\pi x}{L}\Bigg)\Bigg]B
\end{aligned} \tag{16}$$

But,

$$f(x,t)=\left(\frac{1}{\epsilon}\right)\left[-MB-\left(\begin{array}{l}\sum_{j=1}^{\infty}\ddot{q}_j(t)\sin\frac{j\pi x}{L}\\+2(\ddot{x}+a_n t)\left(\frac{j\pi}{L}\right)\sum_{j=1}^{\infty}\dot{q}_j(t)\cos\frac{j\pi x}{L}\\+a_n\left(\frac{j\pi}{L}\right)^2\sum_{j=1}^{\infty}j^2 q_j(t)\sin\frac{j\pi x}{L}\end{array}\right)B\right] \tag{17}$$

Equation 17 becomes:

$$\begin{aligned}f(x,t)=\left(\frac{1}{\epsilon}\right)\Bigg[-M\Bigg(g(B)-\Bigg(&\sum_{j=1}^{\infty}\ddot{q}_j(t)\sin\frac{j\pi x}{L}\\&+2(\ddot{x}+a_n t)\left(\frac{j\pi}{L}\right)\sum_{j=1}^{\infty}\dot{q}_j(t)\cos\frac{j\pi x}{L}\\&+a_n\left(\frac{j\pi}{L}\right)^2\sum_{j=1}^{\infty}j^2 q_j(t)\sin\frac{j\pi x}{L}\Bigg)B\Bigg]\end{aligned} \tag{18}$$

Let

$$f(x,t)=\sum_{j=1}^{\infty}q_{fj}(t)\sin\frac{j\pi x}{L} \tag{19}$$

Also, the applied force can be expressed as a series similar to Equation 12:

$$\begin{aligned}\sum_{j=1}^{\infty}q_{fj}(t)\sin\frac{j\pi x}{L}=\left(\frac{1}{\epsilon}\right)\Bigg[(-MgB)-\Bigg(&\sum_{j=1}^{\infty}\ddot{q}_j(t)\sin\frac{j\pi x}{L}\\&+2(\ddot{x}+a_n t)\left(\frac{j\pi}{L}\right)\sum_{j=1}^{\infty}\dot{q}_j(t)\cos\frac{j\pi x}{L}\\&+a_n\left(\frac{j\pi}{L}\right)^2\sum_{j=1}^{\infty}j^2 q_j(t)\sin\frac{j\pi x}{L}\Bigg)B\Bigg]\end{aligned} \tag{20}$$

Multiply Equation 20 by $\sin\frac{i\pi x}{L}$ and integrate, we have

$$\begin{aligned}\sum_{j=1}^{\infty} q_{fj}(t)\int_0^L \sin\frac{i\pi x}{L}\sin\frac{j\pi x}{L}dx \\ &= -\frac{M}{\epsilon}\left(\int_0^L \sin\frac{i\pi x}{L}B\right)dx \\ &+ \frac{M}{\epsilon}\left(\sum_{j=1}^{\infty}\ddot{q}_j(t)\int_0^L \sin\frac{i\pi x}{L}\sin\frac{j\pi x}{L}B\right)dx \\ &- \frac{2M}{\epsilon}L_{11}\sum_{j=1}^{\infty}\dot{q}_j(t)\int_0^L \sin\frac{i\pi x}{L}\cos\frac{j\pi x}{L}Bdx \\ &- \frac{M}{\epsilon}L_{12}\sum_{j=1}^{\infty} j^2 q_j(t)\int_0^L \sin\frac{i\pi x}{L}\sin\frac{j\pi x}{L}Bdx\end{aligned} \tag{21}$$

where L_{11} represent $(\ddot{x}+a_n t)\left(\frac{j\pi}{L}\right)$.

L_{22} represent $a_n\left(\frac{\pi}{L}\right)^2$, i.e., $Z_{15} = Z_{11} + Z_{12} + Z_{13} + Z_{14}$ where:

$$Z_{11} = -\frac{Mg}{\epsilon}\int_0^L \sin\frac{i\pi x}{L}Bdx \tag{22a}$$

$$Z_{12} = -\frac{M}{\epsilon}\left(\sum_{j=1}^{\infty}\ddot{q}_j(t)\int_0^L \sin\frac{i\pi x}{L}\sin\frac{j\pi x}{L}B\right)dx \tag{22b}$$

$$Z_{13} = -\frac{2M}{\epsilon}L_{11}\sum_{j=1}^{\infty}\dot{q}_j(t)\int_0^L \sin\frac{i\pi x}{L}\cos\frac{j\pi x}{L}Bdx \tag{22c}$$

$$Z_{14} = \frac{M}{\epsilon}L_{12}\sum_{j=1}^{\infty} j^2 q_j(t)\int_0^L \sin\frac{i\pi x}{L}\sin\frac{j\pi x}{L}Bdx \tag{22d}$$

and

$$Z_{15} = \sum_{j=1}^{\infty} q_{fj}(t)\int_0^L \sin\frac{i\pi x}{L}dx \tag{22e}$$

In evaluating the integrals of Equations 22(a) to 22(e), we made use of the following

$$H\left(x-\left(\xi-\frac{\epsilon}{2}\right)\right)=\begin{cases}0, & if\ \ x<\xi-\frac{\epsilon}{2}\\ 1, & if\ \ x>\xi-\frac{\epsilon}{2}\end{cases}$$

$$H\left(x-\left(\xi+\frac{\epsilon}{2}\right)\right)=\begin{cases}0, & if\ \ x<\xi+\frac{\epsilon}{2}\\ 1, & if\ \ x>\xi-\frac{\epsilon}{2}\end{cases}$$

By making use of the Dirac delta function, Equations 22(a) to 22(e) becomes:

$$Z_{11}=-\frac{Mg}{\epsilon}\int_0^L \sin\frac{i\pi}{L}\left(\xi+\frac{\epsilon}{2}\right)d\epsilon-\sin\frac{i\pi}{L}\left(\xi-\frac{\epsilon}{2}\right)d\epsilon \tag{23a}$$

$$\begin{aligned}Z_{12}=-\frac{M}{\epsilon}\sum_{j=1}^{\infty}\ddot{q}_j(t)\int_0^L&\left[\sin\frac{i\pi}{L}\left(\xi+\frac{\epsilon}{2}\right)\sin\frac{i\pi}{L}\left(\xi-\frac{\epsilon}{2}\right)\right.\\ &\left.-\sin\frac{j\pi}{L}\left(\xi+\frac{\epsilon}{2}\right)\sin\frac{j\pi}{L}\left(\xi-\frac{\epsilon}{2}\right)\right]d\epsilon\end{aligned} \tag{23b}$$

$$\begin{aligned}Z_{13}=-\frac{2M}{\epsilon}L_{11}\sum_{j=1}^{\infty}\dot{q}_j(t)\int_0^L&\left[\sin\frac{i\pi}{L}\left(\xi+\frac{\epsilon}{2}\right)\right.\\ &\left.-\sin\frac{i\pi}{L}\left(\xi-\frac{\epsilon}{2}\right)\cos\frac{j\pi}{L}\left(\xi+\frac{\epsilon}{2}\right)-\cos\frac{j\pi}{L}\left(\xi-\frac{\epsilon}{2}\right)\right]d\epsilon\end{aligned} \tag{23c}$$

$$\begin{aligned}Z_{14}=-\frac{M}{\epsilon}L_{12}\sum_{j=1}^{\infty}j^2q_j(t)&\left[\sin\frac{i\pi}{L}\left(\xi+\frac{\epsilon}{2}\right)\right.\\ &\left.-\sin\frac{i\pi}{L}\left(\xi-\frac{\epsilon}{2}\right)\sin\frac{j\pi}{L}\left(\xi+\frac{\epsilon}{2}\right)\sin\frac{j\pi}{L}\left(\xi-\frac{\epsilon}{2}\right)\right]d\epsilon\end{aligned} \tag{23d}$$

and

$$Z_{15}=q_{fj}(t) \tag{23e}$$

Evaluating the integrals in Equations 23(a) to 23(e), we finally obtained:

$$\begin{aligned} q_{fj}(t) = & -\frac{2MgL}{j\pi\epsilon}\sin\frac{i\pi\xi}{L}\sin\frac{i\pi\epsilon}{2L} \\ & -\frac{2M}{j\pi\epsilon}\sum_{j=1}^{\infty}\ddot{q}_j(t)\frac{1}{b_{12}}\left[\sin\frac{i\pi\xi}{L}b_{12}\cos\frac{\pi\epsilon}{L}b_{12}\right]-\frac{1}{b_{12}}\left(\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right) \\ & -\frac{2M}{\epsilon}L_{11}\sum_{j=1}^{\infty}\dot{q}_j(t)\left[\sin\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}+\frac{1}{b_{12}}\sin\frac{\pi\xi}{L}b_{12}\sin\frac{\pi\epsilon}{2L}b_{12}\right] \\ & -\frac{M}{\epsilon}L_{12}\sum_{j=1}^{\infty}q_j(t)\frac{1}{b_{11}}\left[\sin\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}+\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right] \end{aligned} \tag{24}$$

Substituting Equations 23(a) to 23(e), and Equation 24 into Equation 16:

$$\begin{aligned} & EIL_{13}\sum_{j=1}^{\infty}q_j(t)\sin\frac{j\pi x}{L}+m\sum_{j=1}^{\infty}\ddot{q}_j(t)\sin\frac{j\pi x}{L}- \\ & \rho_0 L_{14}\sum_{j=1}^{\infty}q_j(t)\sin\frac{j\pi x}{L}+K\sum_{j=1}^{\infty}q_j(t)\sin\frac{j\pi x}{L} \\ & \quad =\sum_{j=1}^{\infty}\sin\frac{j\pi x}{L}\left[-\frac{2MgL}{j\pi\epsilon}\sin\frac{i\pi\xi}{L}\sin\frac{i\pi\epsilon}{2L}\right. \\ & \quad -\frac{2M}{j\pi\epsilon}\sum_{j=1}^{\infty}\ddot{q}_j(t)\frac{1}{b_{12}}\left[\sin\frac{i\pi\xi}{L}b_{12}\cos\frac{\pi\epsilon}{L}b_{12}\right]-\frac{1}{b_{12}}\left(\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right) \\ & \quad -\frac{2M}{\epsilon}L_{11}\sum_{j=1}^{\infty}\dot{q}_j(t)\left[\sin\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}+\frac{1}{b_{12}}\sin\frac{\pi\xi}{L}b_{12}\sin\frac{\pi\epsilon}{2L}b_{12}\right] \\ & \quad \left.-\frac{M}{\epsilon}L_{12}\sum_{j=1}^{\infty}q_j(t)\frac{1}{b_{11}}\left[\sin\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}+\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right]\right] \end{aligned} \tag{25}$$

where L_{13} represents $\left(\frac{\pi}{L}\right)^4$, L_{14} represents $\left(\frac{\pi}{L}\right)^2$, b_{11} represents $i + j$ and b_{12} represents $i - j$. Simplifying Equation 25, we have:

$$\sum_{j=1}^{\infty}\Bigg[\Big[EIL_{13}qj(t)+m\ddot{q}_j(t)-\rho_0 L_{14}qj(t)+Kqj(t)-\Bigg[-\frac{2MgL}{j\pi\epsilon}\sin\frac{i\pi\xi}{L}\sin\frac{i\pi\epsilon}{2L}$$
$$-\frac{2M}{\epsilon\pi}\sum_{j=1}^{\infty}\ddot{q}_j(t)\frac{1}{b_{12}}\left[\sin\frac{i\pi\xi}{L}b_{12}\cos\frac{\pi\epsilon}{L}b_{12}\right]-\frac{1}{b_{12}}\left(\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right)$$
$$-\frac{2M}{\epsilon}L_{11}\sum_{j=1}^{\infty}\dot{q}_j(t)\left[\sin\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}+\frac{1}{b_{12}}\sin\frac{\pi\xi}{L}b_{12}\sin\frac{\pi\epsilon}{2L}b_{12}\right] \tag{26}$$
$$-\frac{M}{\epsilon}L_{12}\sum_{j=1}^{\infty}q_j(t)\frac{1}{b_{11}}\Bigg[\sin\frac{\pi\xi}{L}b_{11}\cos\frac{\pi\epsilon}{2L}b_{11}$$
$$+\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\Bigg]\Bigg]\Bigg]\sin\frac{j\pi x}{L}=0$$

$$j=1,2,3,\ i\neq j$$

which implies that

$$\begin{aligned}
&m\ddot{q}_j(t)+\left(EIL_{13}+m-\rho_0 L_{14}\right)q_j(t)\\
&\qquad=-\frac{2MgL}{j\pi\epsilon}\sin\frac{i\pi\xi}{L}\sin\frac{i\pi\epsilon}{2L}-\frac{2M}{\epsilon\pi}\sum_{j=1}^{\infty}\ddot{q}_j(t)\frac{1}{b_{12}}\left[\sin\frac{i\pi\xi}{2L}b_{12}\cos\frac{\pi\epsilon}{L}b_{12}\right]\\
&\qquad-\frac{1}{b_{12}}\left(\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right)\\
&\qquad-\frac{2M}{\epsilon}L_{11}\sum_{j=1}^{\infty}\dot{q}_j(t)\left[\sin\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}+\frac{1}{b_{12}}\sin\frac{\pi\xi}{L}b_{12}\sin\frac{\pi\epsilon}{2L}b_{12}\right]\\
&\qquad-\frac{M}{\epsilon}L_{12}\sum_{j=1}^{\infty}q_j(t)\frac{1}{b_{11}}\left[\sin\frac{\pi\xi}{L}b_{11}\cos\frac{\pi\epsilon}{2L}b_{11}+\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right]
\end{aligned} \tag{27}$$

Rearranging Equation 27, we have:

$$
\begin{aligned}
& m\ddot{q}_j(t) + L_{15} q_j(t) \\
&\quad = -\frac{2MgL}{j\pi\epsilon}\sin\frac{i\pi\xi}{L}\sin\frac{i\pi\epsilon}{2L} \\
&\quad -\frac{2M}{\epsilon\pi}\sum_{j=1}^{\infty}\ddot{q}_j(t)\frac{1}{b_{12}}\left[\sin\frac{i\pi\xi}{2L}b_{12}\cos\frac{\pi\epsilon}{L}b_{12}\right] - \frac{1}{b_{12}}\left(\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right) \\
&\quad -\frac{2M}{\epsilon}L_{11}\sum_{j=1}^{\infty}\dot{q}_j(t)\left[\sin\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11} + \frac{1}{b_{12}}\sin\frac{\pi\xi}{L}b_{12}\sin\frac{\pi\epsilon}{2L}b_{12}\right] \\
&\quad -\frac{M}{\epsilon}L_{12}\sum_{j=1}^{\infty}q_j(t)\frac{1}{b_{11}}\left[\sin\frac{\pi\xi}{L}b_{11}\cos\frac{\pi\epsilon}{2L}b_{11} + \cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right]
\end{aligned}
\tag{28}
$$

where $L_{15} = EIL_{13} + m - \rho_0 L_{14}$.

3. NUMERICAL RESULTS

Numerical computation has been carried out by using the central differential technique. Applying this numerical method to derivatives in Equation 29, we obtain:

$$
\begin{aligned}
& m\left(\frac{q_{j+1} - 2q_j + q_{j-1}}{h^2}\right) + L_{15} q_j \\
&\quad = -\frac{2MgL}{j\pi\epsilon}\sin\frac{i\pi\xi}{L}\sin\frac{i\pi\epsilon}{2L} - \frac{2M}{\epsilon\pi}\left(\frac{q_{b_{11}} - 2q_j + qb_{12}}{h^2}\right) \\
&\quad \frac{1}{b_{12}}\left[\sin\frac{i\pi\xi}{2L}b_{12}\cos\frac{\pi\epsilon}{L}b_{12} - \frac{1}{b_{12}}\left(\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right)\right] \\
&\quad -\frac{2M}{\epsilon}L_{11}\left(\frac{q_{b_{11}} - q_{b_{12}}}{2h}\right)\left[\sin\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11} + \frac{1}{b_{12}}\sin\frac{\pi\xi}{L}b_{12}\sin\frac{\pi\epsilon}{2L}b_{12}\right] \\
&\quad -\frac{M}{\epsilon}L_{12}q_j\frac{1}{b_{11}}\left[\sin\frac{\pi\xi}{L}b_{11}\cos\frac{\pi\epsilon}{2L}b_{11} + \cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right]
\end{aligned}
\tag{29}
$$

$j = 1,2,3, \quad i \neq j$

Simplifying Equation 29, we finally obtain:

$$
\begin{aligned}
&-\frac{2MgLh^2}{j\pi\epsilon}\sin\left(\frac{i\pi\xi}{L}\right)\sin\left(\frac{i\pi\epsilon}{2L}\right)\\
&\quad=\left[m+\frac{2M}{\epsilon\pi}\frac{1}{b_{12}}\left[\sin\frac{i\pi\xi}{2L}b_{12}\cos\frac{\pi\epsilon}{L}b_{12}-\frac{1}{b_{12}}\left(\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right)\right]\right.\\
&\quad\left.-\frac{2M}{\epsilon}L_{11}h\left[\sin\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}+\frac{1}{b_{12}}\sin\frac{\pi\xi}{L}b_{12}\sin\frac{\pi\epsilon}{2L}b_{12}\right]\right]q_{b_{11}}\\
&\quad+\left[\left(-2\left(m+h^2L_{15}\right)-\frac{4mj}{\epsilon\pi}\frac{1}{b_{12}}\left(\sin\frac{\pi\xi}{L}b_{12}\cos\frac{\pi\epsilon}{2L}b_{12}\right)\right.\right.\\
&\quad\left.-\frac{1}{b_{11}}\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right)-\frac{m}{\epsilon}L_{12}\frac{1}{b_{11}}\left(\sin\frac{\pi\xi}{L}b_{11}\cos\frac{\pi\epsilon}{2L}b_{11}\right.\\
&\quad\left.\left.+\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right)\right]q_j+\left[m+\frac{2M}{\epsilon\pi}\frac{1}{b_{12}}\left(\sin\frac{\pi\xi}{L}b_{12}\cos\frac{\pi\epsilon}{2L}b_{12}\right)\right.\\
&\quad+\frac{1}{b_{11}}\left(\cos\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right)+\frac{1}{b_{11}}\frac{mL_{11}hj}{\epsilon\pi}\left(\sin\frac{\pi\xi}{L}b_{11}\sin\frac{\pi\epsilon}{2L}b_{11}\right.\\
&\quad\left.\left.+\frac{1}{b_{11}}\sin\frac{\pi\xi}{L}b_{12}\sin\frac{\pi\epsilon}{2L}b_{12}\right)\right]q_{b_{12}}
\end{aligned}
\tag{30}
$$

$j = 1,2,3, \quad i \neq j$

4. DISCUSSION OF RESULTS

The following parameters were used for the numerical analysis: $EI = 1.74 \times 10^{-5}$, $L = 10$ m, $j = 2$, $I = 2$, $\rho_0 = 5$, $\xi = 0.12$, $\epsilon = 0.1$, $v = 3.3$ ms^{-1}, $M = 50$ kg, $m = 984$ kg. Figures 2(a) and 2(b) show the deflection of beam for different values of the mass of the load (M) for which the acceleration of the moving load is considered and neglected, respectively. It is observed that the deflection is increased by increasing the values of the mass of the load. It is seen that the values of the deflection with acceleration being considered is higher than the system where acceleration of the moving load is negligible. If acceleration increases, it causes higher deflection where acceleration is being considered.

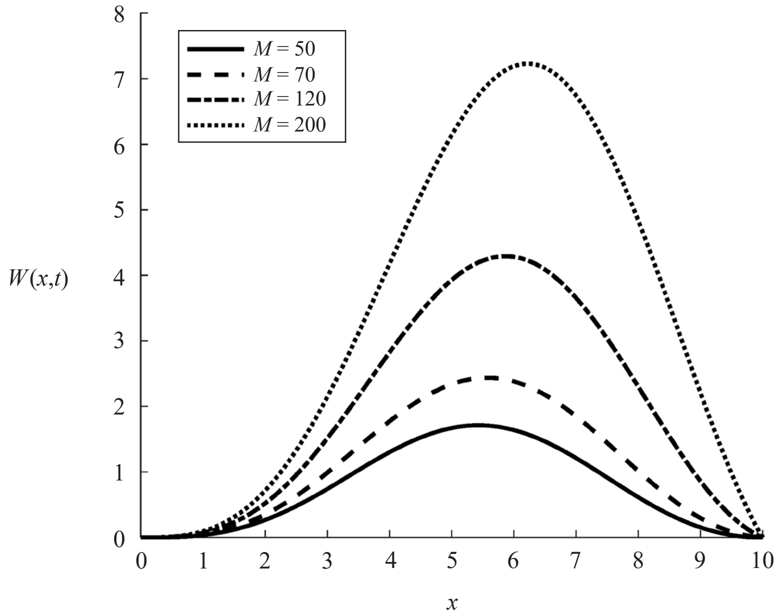

Figure 2(a): Deflection of beam at various values of the *M* (acceleration considered).

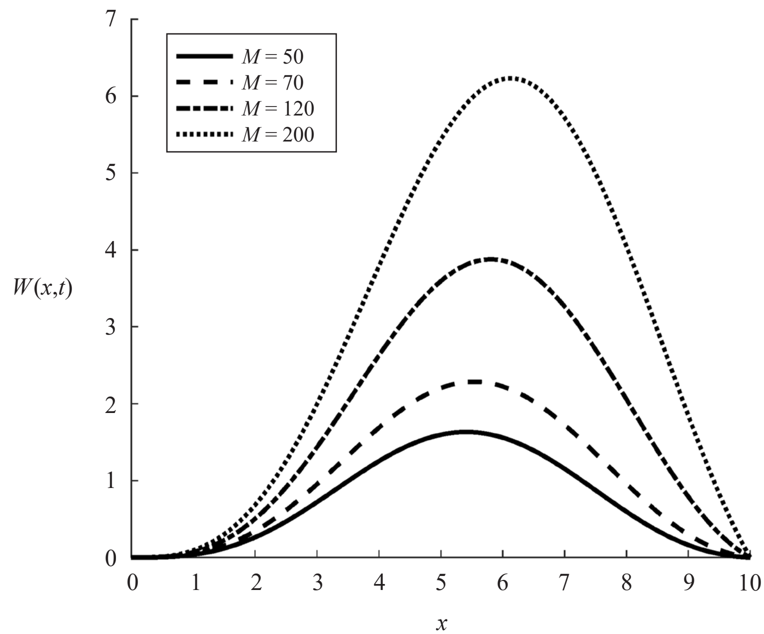

Figure 2(b): Deflection for beam at various values of the *M* (acceleration neglected).

Figures 3(a) and 3(b) show the deflection of beam for different values of the speed of the moving load (v) for which the acceleration of the v is considered and neglected, respectively. It is observed that the deflection increases as the values of the speed of the v is increasing. It is seen that the values of the deflection with acceleration being considered is higher than the system where acceleration of the v is negligible.

Figures 4(a) and 4(b) show the deflection of beam for different values of the length of the load (ξ) for which the acceleration of the v is considered and neglected, respectively. It is observed that the deflection is increased by increasing the values of ξ. It is seen that the values of the deflection with acceleration being considered is higher than the system where acceleration of the v is negligible.

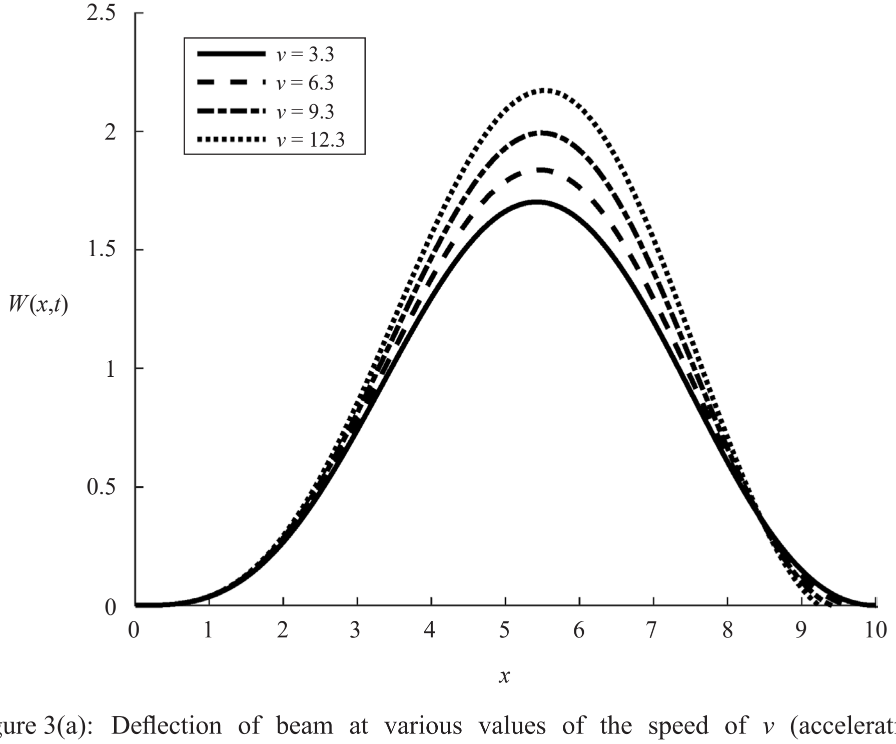

Figure 3(a): Deflection of beam at various values of the speed of v (acceleration considered).

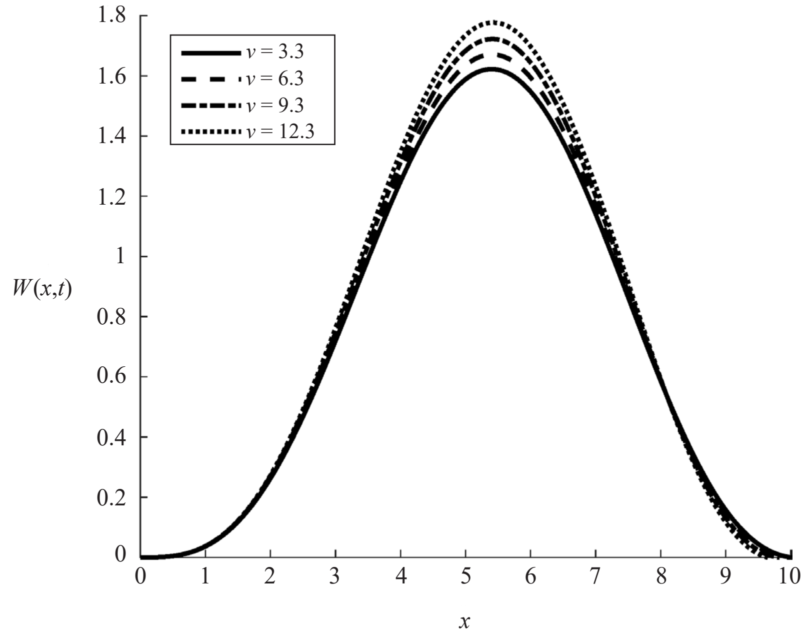

Figure 3(b): Deflection of beam for various values of the speed of v (acceleration neglected).

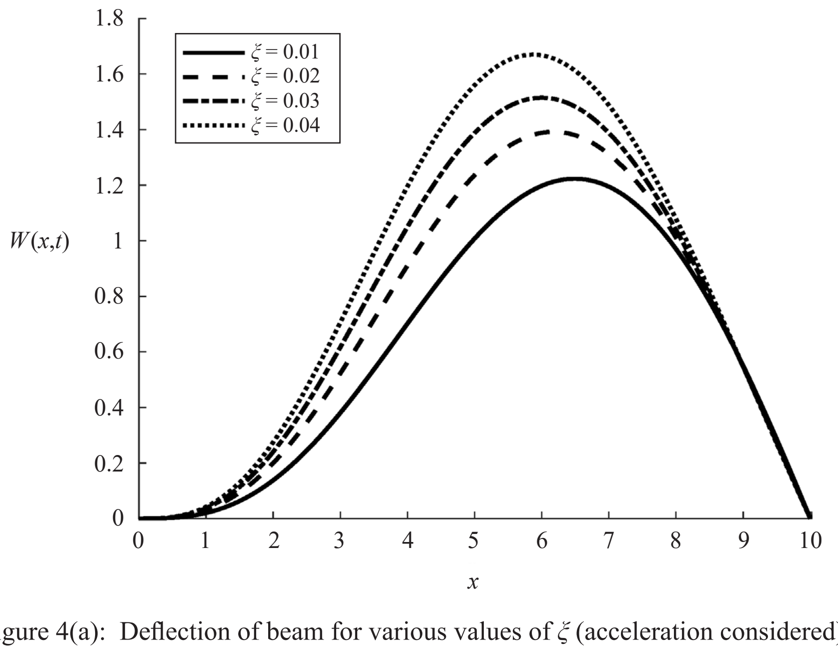

Figure 4(a): Deflection of beam for various values of ξ (acceleration considered).

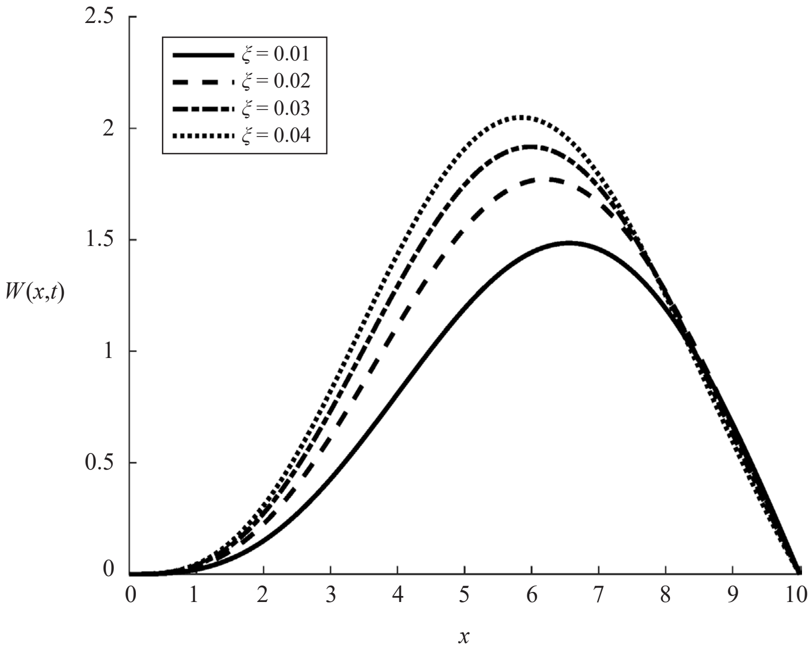

Figure 4(b): Deflection of beam for various values of ξ (acceleration neglected).

Figures 5(a) and 5(b) show the deflection of beam for different values of the distance moved of the load (ϵ) for which the acceleration of v is considered and neglected, respectively. It is observed that the deflection increases as the values of ϵ is increasing. It is seen that the values of the deflection with acceleration being considered is higher than the system where acceleration of v is negligible.

Figures 6(a) and 6(b) show the deflection of beam for different values of the density (ρ_0) when the acceleration of v is not considered. The deflection of the beam increases as the value of mass per unit length increases. For some mass ratios, if acceleration increases, it causes higher deflection for some acceleration values such as 4 ms^{-2} when the mass reaches approximately 60% of the total length of the beam. The deflection curve shows a kind of resonance.

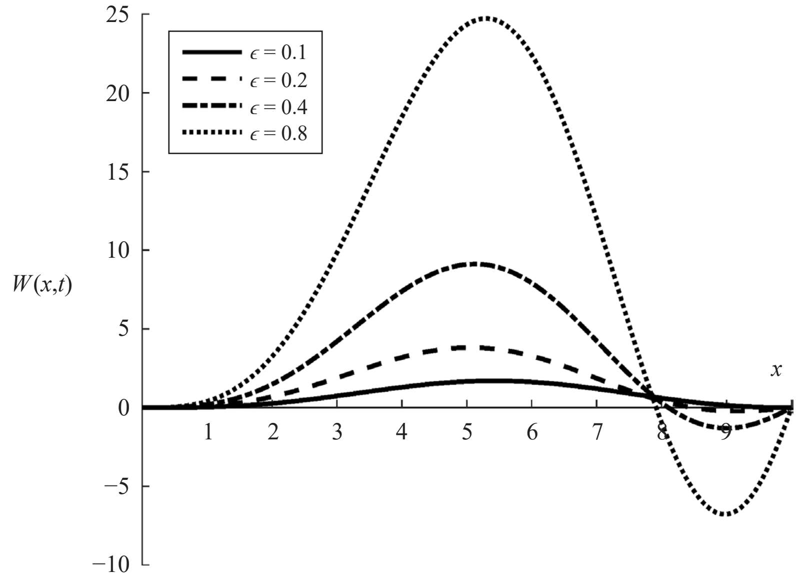

Figure 5(a): Deflection of beam for various values of ϵ (acceleration considered).

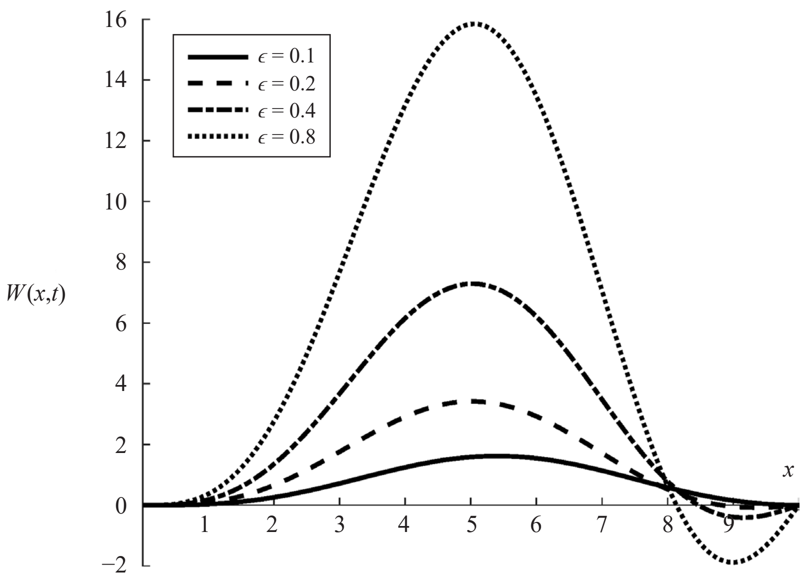

Figure 5(b): Deflection of beam for various values of ϵ (acceleration neglected).

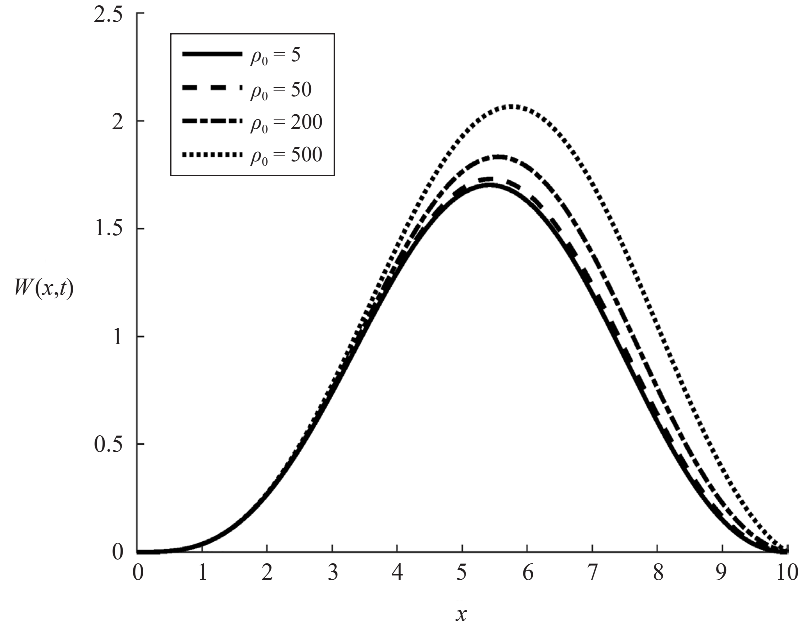

Figure 6(a): Deflection of beam for various values of ρ_0 (acceleration considered).

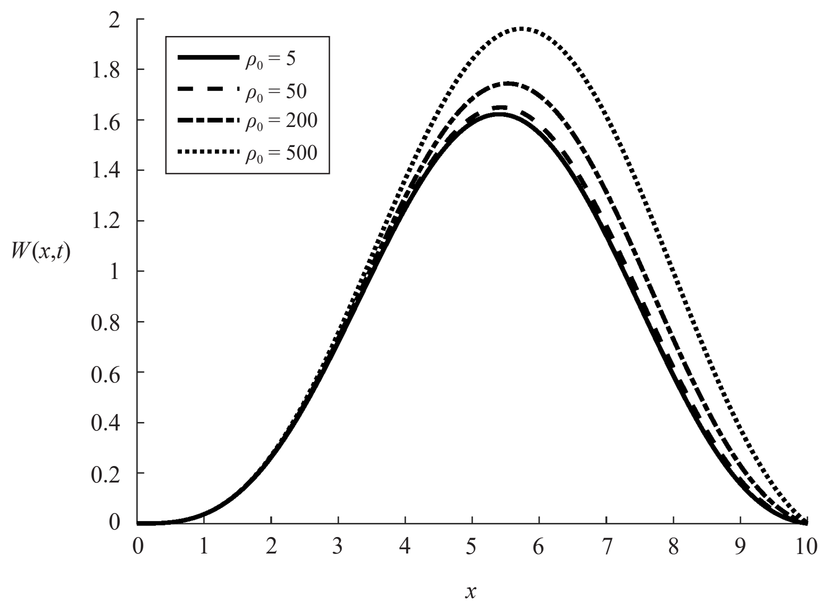

Figure 6(b): Deflection of beam for various values of ρ_0 (acceleration neglected).

In order to validate the efficiency of the method presented and the data used, the results are compared with those available in the literature. In addition, various parameters used are varied to see the dynamic response of the beam and the result were given in graphical form. It is observed that under uniform velocity type of motion, the deflection increases as the velocity increases until it gets to a certain point and start decreasing.

5. CONCLUDING REMARKS

In this article, analytical solution for the BVP for Euler-Bernoulli beam subjected to accelerated load is investigated. It is observed from the figures that for accelerated and uniform velocity type of motion, the deflection is higher in case of moving force than that of the moving mass system. For the decelerated motion, the deflection is higher in the case of moving mass than that of the moving force system. Acceleration of a travelling mass over a structural system, highly affects the dynamic response of the structural system. This will give the engineers some advantages to make a more realistic modelling of structural systems under accelerating mass motion than the classical methods that omit the initial effects of accelerating mass.

6. APPENDIX

Description of parameters:

Parameters	Description
E	Young modulus
I	Inertia moment of the cross section
m	Mass of the beam
M	Mass of the load
x	Beam centre coordinate
t	Time
$W(x,t)$	Vertical deflection of the beam
g	Acceleration due to gravity
v	Velocity of the moving load
ρ	Mass per unit length
$f(x,t)$	Applied moving load
K or k	Winkler foundation

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